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The Asymmetric Avalanche Process

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An asymmetric stochastic process describing the avalanche dynamics on a ring is proposed. A general kinetic equation which incorporates the exclusion and avalanche processes is considered. The Bethe ansatz method is used to calculate the generating function for the total distance covered by all particles. It gives the average velocity of particles which exhibits a phase transition from an intermittent to continuous flow. We calculated also higher cumulants and the large deviation function for the particle flow. The latter has the universal form obtained earlier for the asymmetric exclusion process and conjectured to be common for all models of the Kardar–Parisi–Zhang universality class.

KEY WORDS: Asymmetric exclusion process; avalanche process; Bethe ansatz; sandpile model.

1. INTRODUCTION

Interacting particle systems with stochastic dynamics[(1)] and particularly the one-dimensional asymmetric exclusion process (ASEP) have been intensively studied,[(2–4)] due to connections to growth processes,[(5)] traffic flows,[(6)] the noisy Burgers equation[(7)] and the Kardar–Parisi–Zhang (KPZ) equation.[(8)] Being one of the simplest examples of integrable non-equilibrium systems, the ASEP allows calculation of some dynamical properties,[(9, 10)] a large deviation function,[(11)] and conditional probabilities.[(12)]

In a standard formulation,[(13)] particles move in such a way that there is always at most one particle per site on the one-dimensional lattice. Every particle hops to its right or left with biased probabilities provided the target site is empty. Otherwise, it does not move. Using a traffic terminology, this kind of interaction between particles can be called a "soft braking."

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Another kind of interaction is an "aggressive braking,"[14, 15] when a particle pushes the particle in front of it and then stops. The displaced particle shifts the next particle in front, if any, and so on. As a result, a chain of adjacent particles is shifted by one lattice space left or right at the same moment of time. Despite apparent non-locality of dynamics, the Bethe ansatz method is still applicable and the resulting Bethe equations are solvable parallel to the ASEP case. Further generalizations of the ASEP have been proposed.[16, 17] In every case, however, an elementary motion of a particle produces a deterministic reconstruction (local or non-local) of the preceding lattice state.

The beginning of intensive study of the ASEP nearly coincides with a burst of interest to the threshold dynamics and avalanche processes. Appeared originally in the sandpile model of self-organized criticality,[18] the avalanche processes have been shown to be related to many different phenomena ranging from an interface depinning to earthquakes.[19]

As an example of the threshold dynamics one can again consider ASEP-like stochastic model at the one dimensional lattice. In this case, however, we admit multiple occupation of a lattice site by particles. Like in the ASEP, each particle hops to its right or left. If the number of particles n at given site exceeds some critical value n_c, the site is unstable and must relax immediately. The relaxation consists in spilling of $m \leqslant n$ particles from the given site to neighboring sites by a fixed rule. If the neighboring sites become unstable, they relax as well. Thus, an avalanche of relaxations spreads over the lattice. The time interval between beginning and ending of every avalanche is negligible in comparison with characteristic hopping time of a single particle.

Comparing to the ASEP with the aggressive braking, a fundamental difference appears, when the spilling rule is stochastic.[20] In this case, the structure of avalanche becomes complicated. Unstable states may appear randomly even if an underlying structure of the lattice state is regular before an avalanche starts. Then, the distance at which avalanches propagate and the total mass of particles involved in an avalanche are random values described by probabilistic distributions. The configurations of particles in the lattice states before and after an avalanche may differ considerably, and the latter results from the first by a series of stochastic spillings.

Another peculiarity of avalanche dynamics is a specific transition into a totally unstable state, when the density of particles exceeds some critical value and an avalanche never stops in the thermodynamic limit of infinitely large lattice.[21] This transition corresponds to change of the time scale characterizing the system, which can be defined for example as a ratio of system size to average velocity of particles. While for low density, the slow diffusion processes prevail, the fast avalanches dominate above the transition point. The existing of two time scales was shown to be responsible for reaching of the self-organized critical state in systems with avalanche dynamics.

The aim of this paper is to give a mathematical description of one kind of avalanche processes, the one-dimensional asymmetric avalanche process (ASAP).[22] The ASAP is a partially asymmetric diffusion process with the totally asymmetric avalanche propagation. The similar directed one-dimensional stochastic avalanches have been considered in ref. 23, where the asymptotic of avalanche distributions in the self-organized critical state have been calculated exactly for the open lattice in the thermodynamic limit. Instead, we study the ASAP on a ring with a fixed number of particles. In this case, the critical value of the density exists, depending on spilling probabilities, which corresponds to the transition from the intermittent to continuous flow when the fast avalanche dynamics becomes dominating. In this paper, we concentrate on dynamical properties of the ASAP below this point.

One of the reasons for the intensive interest to the ASEP is that it being exactly solvable gives a discrete version of the Kardar–Parisi–Zhang (KPZ) equation.[10] In the scaling limit, one can get analytically the universal quantities like critical exponents and scaling functions characterizing a vast class of nonequilibrium phenomena belonging to the KPZ universality class. On the other hand, the universal scaling properties of avalanche dynamics are much less investigated. There are very few successful attempts,[24–28] to find analytical arguments allowing one to relate the avalanche-like processes with one of well-defined universality classes such as the KPZ, Edwards–Wilkinson or directed percolation.[29]

In the present work, we show that the generating function of the total distance Y_t travelled by particles in the ASAP is given in the scaling limit by the expression

$$\lim_{t\to\infty} \frac{\ln \langle e^{\gamma Y_t}\rangle}{t} \sim \gamma K_1 + K_2 G(K_3\gamma), \tag{1.1}$$

where $G(x)$ does not depend on parameters of the model and has the following parametric form

$$G(x) = -\sum_{s=1}^{\infty} (-C)^s s^{-5/2} \tag{1.2}$$

$$x = \sum_{s=1}^{\infty} (-C)^s s^{-3/2}, \tag{1.3}$$

and K_1, K_2, and K_3 are model dependent parameters. This universal form of function $G(x)$ was claimed to be the feature of the KPZ universality class.[30, 31] It also determines the universal form of the large deviation function characterizing the deviations of the integrated particle current from its average value. These results give an evidence that the ASAP shares the KPZ universality class with the ASEP despite the significant difference in their dynamics.

The article is organized as follows. In Section 2, we consider the master equation for a general stochastic model which leads to the ASEP and ASAP in particular cases. In Section 3, we derive the Bethe ansatz equations for the generating function of the total displacement of all particles. A particular case of the ASAP is considered in Section 4 where the ASAP becomes identical to a drop-push version of the ASEP. The general case of the ASAP is considered in Section 5. Using the method developed in refs. 32 and 33, we analyse the integral equation corresponding to the density of roots of the Bethe equations and derive the generating function for the total displacement of particles. From it we obtain the expression for the cumulants of total distance travelled by particles, like mean velocity and variance, its large deviation function, and demonstrate that the ASAP belongs to the KPZ universality class.[3]

2. THE MASTER EQUATION

In this section we are going to obtain the master equation describing ASAP, which is defined as follows. Consider the system of p particles on a ring of N sites as shown in Fig. 1. Particles jump left or right with probabilities Ldt or Rdt, respectively, for infinitesimal time dt independently of each other. When a particle comes to already occupied site after the hopping either from left with the rate R or from right with the rate L, an avalanche starts. It develops step by step according to the following dynamical rules.

If, at some step of the avalanche, $n(n=2, 3,...)$ particles are at site x, then

with probability μ_n, n particles go to the site $x+1$;

with probability $1-\mu_n$, $n-1$ particles go to the site $x+1$ and one particle stays at the current site x.

We imply that an avalanche takes infinitesimal time to end, i.e., from the point of view of Poissonian processes it plays a role of interaction resulting in the transition between configurations with single particle

[3] A brief summary of the results has been presented at "StatPhys-Taiwan 2002," during May 27–June 1, see ref. 34.

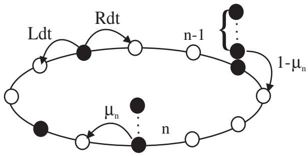

Fig. 1. The asymmetric avalanche process.

occupation. The totally asymmetric case discussed in ref. 22 corresponds to particular choice of the rates, $L=0$, $R=1$. In the case of the ASEP a particle step to already occupied site is forbidden. However, it will be shown below to be closely connected with the ASAP. To make the presentation more systematic we start from the system of free particles then going to ASEP and ASAP dynamics.

The state C of the system at time t is characterized by the probability $P_t(C)$ satisfying the master equation

$$\partial_t P_t(C) = \sum_{\{C'\}} M(C, C')\, P_t(C') \tag{2.1}$$

The off-diagonal elements $M(C, C')$ of the matrix M are rates of transitions from configurations C' to C and therefore are always positive. The diagonal elements $M(C, C)$ that give the total rate of the transition from the state C to all other configurations, enter the matrix M with a minus sign. Conservation of probability requires the identity

$$M(C, C) = -\sum_{\{C'\}} M(C', C) \tag{2.2}$$

Let us return to particles at the lattice. Consider noninteracting particles jumping left or right with probabilities Ldt or Rdt, respectively, for infinitesimal time dt. The probability $P_t(x_1,..., x_p)$ for particles to occupy sites $x_1,..., x_p$ obeys the master equation

$$\begin{aligned}\partial_t P_t(x_1,..., x_p) = -pP_t(x_1,..., x_p) + L \sum_{i=1}^{p} P_t(x_1,..., x_i+1,..., x_p)\\ + R \sum_{i=1}^{p} P_t(x_1,..., x_i-1,..., x_p)\end{aligned} \tag{2.3}$$

if $x_{i+1}-x_i>1$.We impose the condition $L+R=1$ by an appropriate choice of time scale.

In the ASEP, the form of the master equation should be modified if configuration C contains pairs of neighboring occupied sites. If there are only two neighboring sites, $x, x+1$, occupied by particles, the master equation differs from Eq. (2.3) by the extra term $LP_t(\ldots, x+1, x+1,\ldots)+RP_t(\ldots, x, x,\ldots)-P_t(\ldots, x, x+1,\ldots)$. If there are more than one pair, one must substract the other unwanted terms from the Eq. (2.3) for every pair to obtain equations taking into account the exclusion rules. Instead, one can reduce the consideration to the free master equation only, if one equates the appeared extra terms to zero putting the boundary conditions for the physical domain $x_1 < x_2 < \cdots < x_p$:

$$LP_t(\ldots, x+1, x+1,\ldots)+RP_t(\ldots, x, x,\ldots)-P_t(\ldots, x, x+1,\ldots)=0. \tag{2.4}$$

The terms like $P_t(\ldots, x, x,\ldots)$ corresponding to multiple occupation of sites do not contribute to the dynamics due to the exclusion rule. Therefore, they can be considered as auxiliary non-physical terms and redefined by the boundary conditions so that all extra terms in free equation vanish giving the correct equation for the system with interaction.

Consider now more general condition of type Eq. (2.4) with the coefficients α and β which do not coincide necessarily with the rates L and R in Eq. (2.3) and will be defined later,

$$\alpha P_t(\ldots, x, x,\ldots)+\beta P_t(\ldots, x+1, x+1,\ldots)-P_t(\ldots, x, x+1,\ldots)=0 \tag{2.5}$$

To provide the probabilistic meaning of the Eq. (2.3) together with Eq. (2.5), some constraints should be imposed on α and β. In this case, we still use the exclusion rule that allows one to use the terms of type $P_t(\ldots, x, x,\ldots)$ as auxiliary ones which should be redefined in appropriate way. The condition Eq. (2.5) itself does not eliminate the contribution of extra terms yet. Nevertheless, we can try to use this condition to replace the unwanted terms by terms consisting of allowed configurations only. To this end, we can exploit the fact that two unphysical terms in Eq. (2.5) are of similar structure and consider this relation as a recursion:

$$P_t(\ldots, x, x,\ldots)=\frac{1}{\alpha}P_t(\ldots, x, x+1,\ldots)-\frac{\beta}{\alpha}P_t(\ldots, x+1, x+1,\ldots), \tag{2.6}$$

To proceed, it is convenient to consider the two-particle case separately.

2.1. The Case of Two Particles

If there are only two particles at the lattice, the recursion Eq. (2.6) can be immediately solved in terms of allowed configurations only:

$$P_t(x, x)=\frac{1}{\beta}\sum_{n=0}^{\infty}\left(-\frac{\alpha}{\beta}\right)^n P_t(x-n-1, x-n) \tag{2.7}$$

where, due to periodic boundary conditions, all coordinates are integers modulo N. Substituting Eq. (2.7) into the Eq. (2.3) rewritten for the two particle case under the condition $x_2 = x_1 + 1$, we get

$$\begin{aligned}\partial_t P_t(x, x+1) = {} & L P_t(x, x+2) + R P_t(x-1, x+1) - P_t(x, x+1) \\ & + \left(\frac{L}{\beta} - 1\right) P_t(x, x+1) \\ & + \frac{L}{\beta}\left(\mu + \frac{R}{L}\right) \sum_{n=1}^{\infty} \mu^{n-1} P_t(x-n, x-n+1)\end{aligned} \tag{2.8}$$

where $\mu = -\alpha/\beta$. To give a probabilistic meaning to the transition rates, the terms corresponding to the processes when the system leaves the configuration $(x, x+1)$ should be non-positive and those for coming into $(x, x+1)$ from other configurations should be non-negative. To keep probabilities positive, we have to impose the condition that either μ is positive or $\mu = -R/L$, when the term containing the infinite sum vanishes. In addition, conservation of probability, Eq. (2.2), requires

$$\alpha + \beta = 1. \tag{2.9}$$

Then, the condition $\mu = -R/L$ implies $\alpha = R$, $\beta = L$, i.e., the ordinary ASEP. In the case $\mu > 0$, we have

$$\alpha = -\mu/(1-\mu), \qquad \beta = 1/(1-\mu) \tag{2.10}$$

and can rewrite Eq. (2.8) in the form

$$\begin{aligned}\partial_t P_t(x, x+1) = {} & L P_t(x, x+2) + R P_t(x-1, x+1) - P_t(x, x+1) \\ & + (R + L\mu)(-P_t(x, x+1) \\ & + (1-\mu) \sum_{n=1}^{\infty} \mu^{n-1} P_t(x-n, x-n+1)\end{aligned} \tag{2.11}$$

In terms of μ, the boundary condition, Eq. (2.6) reads

$$P_t(x, x) = (1-\mu)\, P_t(x-1, x) + \mu P_t(x-1, x-1) \tag{2.12}$$

The expression Eq. (2.11) shows that in addition to the Poissonian hopping given by the original kinetic equation, new terms appear in the equation which correspond to transitions to the configuration $C = (x, x+1)$ from the configurations $\{C'\} = \{(x-n, x-n+1),\ n = 1, 2,...\}$. The rates of the transitions are $(R + L\mu)(1-\mu)\, \mu^{n-1}$. In the case of two particles, these rates determine the avalanche dynamics defined above.

To show the relation between ASEP and ASAP before going into details of the solution, let us consider the moment when an avalanche starts. This happens if either the left particle of two neighboring ones moves right with the rate R or the right particle moves to the left with the rate L and then two particles together make at least one step together with probability μ. Then an avalanche starts with the rate $(R+L\mu)$. This expression indeed enters those parts of kinetic equation, which correspond to the avalanche dynamics. Thus, the rate of beginning of an avalanche becomes zero when $\mu=-R/L$, and only exclusion dynamics remains. Therefore, we may treat ASEP as an analytical continuation of ASAP with a parameter μ taking a special negative value.

2.2. Many Particle Processes

One can expect that the n-particle interactions imposes n new constrains on the master equation Eq. (2.3). However, in this section we will show that under certain constraint on toppling probabilities μ_n no new boundary conditions appear and Eq. (2.12) is sufficient to take into account the interaction of arbitrary number of particles.

To generalize the boundary condition, Eq. (2.12), for the description of many particle dynamics defined above one should express an unstable configuration via configurations appeared at the previous steps of an avalanche. The form of these conditions depends on the fact whether the site $x-1$ is occupied or not:

$$P_t(\dots, x-1, \underbrace{x,\dots, x}_{n-1},\dots) = (1-\mu_n)\, P_t(\dots, \underbrace{x-1,\dots, x-1}_{n},\dots) + (1-\mu_{n-1})\, P_t(\dots, \underbrace{x-1,\dots, x-1}_{n-1}, x,\dots) \tag{2.13}$$

if the site $x-1$ is occupied, and

$$P_t(\dots, \underbrace{x,\dots, x}_{n},\dots) = \mu_n P_t(\dots, \underbrace{x-1,\dots, x-1}_{n},\dots) + \mu_{n-1} P_t(\dots, \underbrace{x-1,\dots, x-1}_{n-1}, x,\dots) \tag{2.14}$$

if $x-1$ is empty.

Like the two-particle boundary condition, Eq. (2.12), the many-particle conditions, Eqs. (2.13) and (2.14), should be applied recursively. Applying this recursion step by step to infinity, we generate an infinite series that consist of the transition probabilities between stable configurations only.

On the other hand, one may treat the term $P_t(\ldots, \underbrace{x,\ldots, x}_{n},\ldots)$ formally applying the two-particle boundary condition Eq. (2.12), reducing sequentially the number of particles in unstable sites. As a result, we obtain

$$P_t(\ldots, \underbrace{x,\ldots, x}_{n},\ldots) = \mu_n P_t(\ldots, \underbrace{x-1,\ldots, x-1}_{n},\ldots)$$
$$+(1-\mu_n)\, P_t(\ldots, \underbrace{x-1,\ldots, x-1}_{n-1}, x,\ldots) \tag{2.15}$$

Due to recursion, parameters μ_n are expressed through the only parameter μ:

$$\mu_2 = \mu, \qquad \mu_3 = (1-\mu)\,\mu, \qquad \mu_n = (1-\mu)\,\mu_{n-1} + \mu\mu_{n-2}, \qquad n > 3 \tag{2.16}$$

or

$$\mu_n = \mu\,\frac{1-(-\mu)^{n-1}}{1+\mu}. \tag{2.17}$$

Generally, Eq. (2.15) and Eqs. (2.13) and (2.14) do not coincide. However, the final series entering the kinetic equation which result from the sequential use of either latter or two former of them will be the same, provided that the transition probabilities μ_n from Eqs. (2.13) and (2.14) governing the avalanche dynamics satisfy Eqs. (2.17) and (2.16).

Indeed, every term of resulting series represents a finite avalanche. It contains the product of terms μ_n and $(1-\mu_n)$ coming from the successive use of the recurrent relations, Eqs. (2.13), (2.14) or Eq. (2.15). The first term of the r.h.s. of Eq. (2.13) increases the number of particles at the unstable site by 1 in comparison with the l.h.s. The second term of the r.h.s. of Eq. (2.14) decreases the number of particles by 1. Obviously, in every finite avalanche, the numbers of decreasing and increasing events are equal. Therefore, the coefficients $(1-\mu_n)$ from Eq. (2.13) and μ_{n-1} from Eq. (2.14) always enter the product corresponding to the avalanche in pairs and interchanging their places do not affect the structure of the final series. On the other hand, the interchanging μ_{n-1} and $(1-\mu_n)$ between Eqs. (2.13) and (2.14) leads to Eq. (2.15) obtained from Eq. (2.12).

Thus, we have shown that the two-particle boundary condition, Eq. (2.12), is sufficient for obtaining the kinetic equation for many particle avalanche process governed by the above dynamical rules. In the Bethe ansatz formalism, a similar procedure is known as the two-particle reducibility[38] and provides integrability of a system.

3. THE BETHE ANSATZ FOR GENERATING FUNCTION

Consider the generating function of the total path Y travelled by all particles between time 0 and t provided that the system is in configuration C at time t

$$F_t(C) = \sum_{Y=-\infty}^{\infty} P_t(C, Y)\, e^{\gamma Y}, \tag{3.1}$$

where $P_t(C, Y)$ is the joint probability that the system is in configuration C and the total distance is Y to the time t. The equation for the generating function and a boundary condition can be obtained from the master equation Eq. (2.3) and the boundary condition for the probability Eq. (2.12) by multiplying by $e^{\gamma}(e^{-\gamma})$ every term corresponding to increasing (decreasing) distance Y by 1.

$$\begin{aligned} \partial_t F_t(x_1,..., x_p) = {} & Le^{-\gamma} \sum_{i=1}^{p} F_t(x_1,..., x_i+1,..., x_p) \\ & + Re^{\gamma} \sum_{i=1}^{p} F_t(x_1,..., x_i-1,..., x_p) - pF_t(x_1,..., x_p) \end{aligned} \tag{3.2}$$

$$\begin{aligned} F_t(..., x, x,...) = {} & (1-\mu)\, e^{\gamma} F_t(..., x-1, x,...) \\ & + \mu e^{2\gamma} F_t(..., x-1, x-1,...). \end{aligned} \tag{3.3}$$

The sum of the function $F_t(C)$ over all configurations gives the generating function of moments of total distance Y_t, whose behavior is determined by the largest eigenvalue $\Lambda(\gamma)$ of Eq. (3.2) for large t.

$$\sum_{\{C\}} F_t(C) = \langle e^{\gamma Y_t} \rangle \sim e^{\Lambda(\gamma)\, t}. \tag{3.4}$$

The derivatives of $\Lambda(\gamma)$ at $\gamma = 0$ give the cumulants of the distance Y_t

$$\lim_{t\to\infty} \frac{\langle Y_t \rangle_c}{t} = \lim_{t\to\infty} \frac{\langle Y_t \rangle}{t} = \left.\frac{\partial \Lambda(\gamma)}{\partial \gamma}\right|_{\gamma=0} \tag{3.5}$$

$$\lim_{t\to\infty} \frac{\langle Y_t^2 \rangle_c}{t} = \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle - \langle Y_t \rangle^2}{t} = \left.\frac{\partial^2 \Lambda(\gamma)}{\partial \gamma^2}\right|_{\gamma=0} \tag{3.6}$$

$$\lim_{t\to\infty} \frac{\langle Y_t^3 \rangle_c}{t} = \lim_{t\to\infty} \frac{\langle Y_t^3 \rangle + 2\langle Y_t \rangle^3 - 3\langle Y_t^2 \rangle \langle Y_t \rangle}{t} = \left.\frac{\partial^3 \Lambda(\gamma)}{\partial \gamma^3}\right|_{\gamma=0} \tag{3.7}$$

The quantity of our main interest is the large deviation function

$$f(y) = \lim_{t \to \infty} \frac{1}{t} \ln \mathrm{Prob}\left(\frac{Y_t}{t} = y\right) \tag{3.8}$$

which characterizes deviations of the distance Y_t from the average value and can be expressed also trough the largest eigenvalue $\Lambda(\gamma)$:

$$f(y) = \left(\Lambda(\gamma) - \gamma \frac{d\Lambda(\gamma)}{d\gamma}\right) - \gamma y \tag{3.9}$$

$$y = \frac{d}{d\gamma} \Lambda(\gamma). \tag{3.10}$$

Thus, we have to find the dependence of $\Lambda(\gamma)$ on the parameter γ. The master equation Eq. (3.2) and the boundary conditions Eq. (3.3) allow one to use the Bethe ansatz in a usual form

$$F_t(x_1,..., x_p) = e^{\Lambda t} \sum_{\sigma(1,..., p)} A(z_{\sigma_1},..., z_{\sigma_p})\, z_{\sigma_1}^{-x_1} \cdots z_{\sigma_p}^{-x_p}, \tag{3.11}$$

where the summation is over all permutations of $(\sigma_1,..., \sigma_p)$. The eigenvalue corresponding to eigenvalue Eq. (3.11) is

$$\Lambda(\gamma) = R \sum_{i=1}^{p} e^{\gamma} z_i + L \sum_{i=1}^{p} \frac{1}{e^{\gamma} z_i} - p. \tag{3.12}$$

The parameters z_i satisfy the Bethe equations

$$z_k^N = (-1)^{p-1} \prod_{j=1}^{p} \frac{1-(1-\mu)\, e^{\gamma} z_k - \mu e^{2\gamma} z_j z_k}{1-(1-\mu)\, e^{\gamma} z_j - \mu e^{2\gamma} z_j z_k} \tag{3.13}$$

which follow from the substitution of Eq. (3.11) into Eq. (3.3) and the periodic boundary conditions. The largest eigenvalue of the master equation corresponds to the stationary state of Markov process, so, one has to choose the solution of Eqs. (3.13) which provides the eigenvalue of the master equation for probability of Eq. (2.3) to be equal to zero

$$\lim_{\gamma \to 0} \Lambda(\gamma) = 0. \tag{3.14}$$

The Perron–Frobenius theorem ensures that this eigenvalue has no crossing in the whole range of γ.

To solve the Bethe equations, it is convenient to transform the variables z_k. After the change of variables

$$z_k = \frac{1-x_k}{1+\mu x_k} e^{-\gamma} \tag{3.15}$$

the system (3.13) can be rewritten in the following way

$$e^{-\gamma N}\left(\frac{1-x_k}{1+\mu x_k}\right)^N = (-1)^{p-1}\prod_{j=1}^{p}\frac{x_k+\mu x_j}{x_j+\mu x_k} \tag{3.16}$$

Corresponding eigenvalue has the form

$$\Lambda(\gamma) = \sum_{k=1}^{p}\left(R\frac{1-x_k}{1+\mu x_k}+L\frac{1+\mu x_k}{1-x_k}\right)-p \tag{3.17}$$

4. THE LIMIT $\mu = 0$

In the limit $\mu \to 0$, the ASAP becomes a particular case of the two-parameter family of exclusion processes discussed in ref. 15 which, in turn, degenerates into the $n=1$ drop-push model[(14)] in the totally asymmetric case $L=0$. In this case, particles perform the partially asymmetric random walk with rates L and R. Going right, a particle jumps to the closest unoccupied site at its right overtaking all adjacent particles next to it, whereas the motion to left obeys the exclusion rule.

The simple form of the Bethe equations in this case allows one to use the method proposed in ref. 11 to obtain a full solution of the problem. If one introduces the parameter

$$B = (-1)^{p-1} e^{-\gamma N}\prod_{j=1}^{p} x_j \tag{4.1}$$

the solution of Eqs. (3.16) will be given by the roots of the polynomial equation

$$B(1-x)^N - x^p = 0. \tag{4.2}$$

To get the largest eigenvalue, one has to choose p roots approaching zero when $\gamma \to 0$. Following ref. 11 one can use the Cauchy theorem to evaluate the sum over these roots integrating along the contour enclosing all the roots in a small vicinity of zero.

$$\sum_{j=1}^{p} f(x_j) = \frac{1}{2\pi i} \oint dx\, f(x) \frac{\frac{N}{1-x} + \frac{p}{x}}{1 - B\frac{(1-x)^N}{x^p}}. \tag{4.3}$$

Inserting $R(1-x)+L/(1-x)-1$ instead of $f(x)$, we get the expression for the eigenvalue in terms of series in powers of B

$$\Lambda(\gamma) = \sum_{k=1}^{\infty} B^k (-1)^{kp}\, C_{Nk-1}^{pk-1} \left(\frac{R}{k(1-\rho)+1/N} - \frac{L(1-\rho)}{k-1/N} \right) \tag{4.4}$$

where $\rho = p/N$ and $C_a^b = a!/(b!(a-b)!)$ is the binomial coefficient. On the other hand, one can get the expression for γ requiring $\prod_{j=1}^{p} z_j = 1$, which is correct for the groundstate solution. Taking the logarithm of this product and using Eq. (4.3) one gets

$$\gamma = \frac{1}{p} \sum_{k=1}^{\infty} \frac{B^k}{k} (-1)^{kp}\, C_{Nk-1}^{pk-1}. \tag{4.5}$$

Resolving two series of Eqs. (4.4), (4.5) and using Eqs. (3.5), (3.6), and (3.7), we obtain the expressions for cumulants of the total distance traveled by particles:

$$\lim_{t\to\infty} \frac{\langle Y_t \rangle_c}{t} = N\rho \left(\frac{R}{(1-\rho)+1/N} - \frac{L(1-\rho)}{1-1/N} \right) \tag{4.6}$$

$$\begin{aligned} \lim_{t\to\infty} \frac{\langle Y_t^2 \rangle_c}{t} = N \frac{\rho^2 C_{2N-1}^{2p-1}}{[C_{N-1}^{p-1}]^2} \Bigg(& \frac{L(1-\rho)}{(1-1/N)(2-1/N)} \\ & + \frac{R}{((1-\rho)+1/N)(2(1-\rho)+1/N)} \Bigg) \end{aligned} \tag{4.7}$$

$$\begin{aligned} \lim_{t\to\infty} \frac{\langle Y_t^3 \rangle_c}{t} = N^2 \rho \Bigg[-3 \frac{[C_{2N-1}^{2p-1}]^2}{[C_{N-1}^{p-1}]^4} \Bigg(& \frac{L(1-\rho)}{(1-1/N)(2-1/N)} \\ & + \frac{R}{((1-\rho)+1/N)(2(1-\rho)+1/N)} \Bigg) \\ + 4 \frac{C_{3N-1}^{3p-1}}{[C_{N-1}^{p-1}]^3} \Bigg(& \frac{L(1-\rho)}{(1-1/N)(3-1/N)} \\ & + \frac{R}{((1-\rho)+1/N)(3(1-\rho)+1/N)} \Bigg) \Bigg] \end{aligned} \tag{4.8}$$

The scaling limit, $N \to \infty$, of these expressions is of interest for us because it provides information about the large scale behavior independent of the details of microscopic dynamics. While the ASEP keeps the same universal behavior for any value of ρ, scaling properties of the ASAP may change depending on how close to critical point the system is. In the case $\mu = 0$, the critical density corresponds to full occupation of the lattice, $\rho_c = 1$. In the subcritical regime which corresponds to

$$(1-\rho) \gg 1/N \tag{4.9}$$

the situation is similar to the ASEP. In the subcritical region, the generating function, $\Lambda(\gamma)$, takes the universal scaling form, Eq. (1.1) which has been already obtained for the ASEP and claimed to be universal for all models of KPZ universality class.[11, 30] Three model-dependent constants K_1, K_2, K_3 in Eq. (1.1) are

$$K_1 = N\rho\left(\frac{R}{(1-\rho)} - L(1-\rho)\right), \tag{4.10a}$$

$$K_2 = N^{-3/2}\sqrt{\frac{\rho}{2\pi(1-\rho)}}\left(\frac{R}{(1-\rho)^2} + L(1-\rho)\right), \tag{4.10b}$$

$$K_3 = N^{3/2}\sqrt{2\pi(1-\rho)\,\rho}. \tag{4.10c}$$

The average velocity of particles,

$$V_\infty = \frac{1}{p}\lim_{t\to\infty}\frac{\langle Y_t\rangle_c}{t} \simeq \frac{R}{(1-\rho)} - L(1-\rho), \tag{4.11}$$

and the other cumulants of the distance travelled by particles become divergent in the thermodynamic limit when ρ approaches 1. However, the physical quantities characterizing the finite system should obviously be finite. As an example, one can consider simultaneous limit $\rho \to 1$, $N \to \infty$. Substituting, for instance, $N^{-\vartheta}$, $(0 < \vartheta < 1)$ instead of $(1-\rho)$ into Eqs. (4.10), (4.11) one gets the expressions for cumulants of Y_t which remain finite for finite N. The velocity of particles in this case, $V_\infty \sim N^{\vartheta}$, becomes explicitly dependent on N. Being finite for finite N, it is divergent when N tends to infinity. The upper limit for the exponent ϑ, given by Eq. (4.9), is due to the term $1/N$ in Eq. (4.4). This term plays the role of "infrared cutoff" at the scale N, which ensures V_∞ to remain of order N if $(1-\rho)$ becomes zero. When $(1-\rho)$ becomes of order of $1/N$, the generating function, $\Lambda(\gamma)$, looses its universal structure, Eq. (1.1). Practically, this means that the presence of characteristic length N breaks the scale invariance specific for the KPZ dynamics:

$$N \to \lambda N, \qquad K_1 \to \lambda K_1, \qquad K_2 \to \lambda^{-3/2} K_2, \qquad K_3 \to \lambda^{3/2} K_3,$$

which is held in the subcritical region, Eq. (4.9). The character of particle motion near the critical line becomes strongly collective. Eventually, in the limit $N = p$, the process is equivalent to totally asymmetric diffusion of a single particle, with the distance, Y_t, and time, t, rescaled as follows

$$Y_t \to Y_t N, \qquad t \to tRN.$$

Although the solution for $\mu = 0$ allows one to approach the vicinity of critical line, it seems to be very specific as it does not, in fact, involve the avalanche dynamics. However, as we will see below, it catches the basic universal scaling properties of the subcritical dynamics of the model for arbitrary μ.

5. THE CASE OF ARBITRARY $\mu < 1$

In the case of arbitrary μ, the Bethe equations Eq. (3.16) cannot be reduced to a polynomial equation. Nevertheless, they still can be solved in the limit $N \to \infty$, $p \to \infty$, $\rho = p/N = \text{const}$. Let us consider the equation obtained by taking the logarithm of both parts of Eq. (3.16)

$$p_0(x_k) - \frac{1}{N} \sum_{j=1}^{p} \Theta(x_j/x_k) - \gamma = 2\pi i Z(x_k) \tag{5.1}$$

$$\Theta(y/x) = \ln x - \ln y + \ln \frac{1 + \mu y/x}{1 + \mu x/y}$$
$$p_0 = \ln\left(\frac{1 - x_k}{1 + \mu x_k}\right) \tag{5.2}$$

We define $\Theta(y/x)$ at the complex plane of variable y with branch cuts shown in Fig. 2. For small positive γ, the solution corresponding to the largest eigenvalue was shown in ref. 30 to behave as

$$x_k \sim r e^{2\pi i \frac{k}{p}}, \qquad r \sim \gamma^{1/p}, \qquad \gamma \to 0 \tag{5.3}$$

The radius r behaves nonanalyticallywhen γ approaches zero, so in the limit $p \to \infty$, radius r becomes finite no matter how γ is close to zero. It can be easily verified that this is also correct for non-zero μ at least in the limit

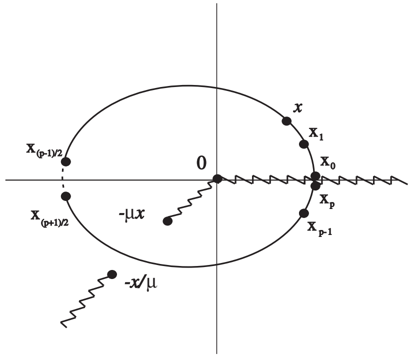

Fig. 2. The analytical structure of the contour Γ. Zigzag lines show the branch cuts of the function $\Theta(y/x)$ in the complex plane of the variable y. The broken segment of the contour should be excluded from integration when $\gamma \neq 0$.

$\gamma = 0$. The analytical function $Z(x)$ is fixed by the choice of logarithm branches. The distribution of roots given by Eq. (5.3) corresponds to the following choice:

$$Z(x_j) = -\frac{1}{N}\left(j - \frac{p+1}{2}\right). \tag{5.4}$$

Then, assuming that the roots are at smooth contour at the complex plane, the derivative of $Z(x)$ with minus sign has a meaning of density of roots along the contour

$$R(x) = -\frac{\partial Z(x)}{\partial x}. \tag{5.5}$$

Instead of Eq. (5.1), we are going to solve the equation

$$p_0(x) - \frac{1}{N}\sum_{j=1}^{p} \Theta(x_j/x) - \gamma = 2\pi i Z(x) \tag{5.6}$$

together with Eqs. (5.4) and (5.5) under the assumption that the density of roots is an analytical function of x. To solve these equations one need to transform Eq. (5.6) to the integral form. This procedure is not straightforward and depends very much on properties of the function $Z(x)$, which should be first assumed and then can be checked a posteriori.

A simplest method is based on the replacement of the sum by the integral along some contour in complex plane in the thermodynamic limit.[35] The analytic solution then can be found for some special cases, particularly for the case of the contour closed around zero.[36] It turns out to be the particular solution of our problem corresponding to single value of γ, $\gamma = 0$. A development of this idea is the expansion method proposed in ref. 37 for the investigation of the conical point of ferroelectric six-vertex model, which allows calculation of the leading term of the first cumulant of Y_t.[22] To obtaine higher cumulants one needs to calculate the finite size corrections to the thermodynamic solution. This has been done for the ASEP[33] with the help of the method of perturbative expansion of the Bethe equations proposed by Kim.[32] For the ASAP we use the modification of his approach, which allows us to calculate the finite size corrections, avoiding some assumptions made in the original method. At least in the leading orders the results do not depend on these assumptions and we reproduce the results by Lee and Kim in the particular case $\mu = -R/L$. To simplify the presentation we leave the details of the solution for the Appendices (A–C), going directly to the results.

The solution of the Bethe equations results in the generating function obtained in the scaling limit $\gamma N^{3/2} = \text{const}$

$$\Lambda(\gamma) = \gamma K_1 + K_2 G(\gamma K_3). \tag{5.7}$$

Here $G(x)$ in a parametric form is defined by the relations

$$G(x) = -\mathrm{Li}_{5/2}(-C), \tag{5.8}$$

$$x = \mathrm{Li}_{3/2}(-C), \tag{5.9}$$

and the function $\mathrm{Li}_k(x)$ is the polylogarithm defined by series

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} \tag{5.10}$$

when $|x| < 1$. For arbitrary negative x, the integral definition can be used:

$$\mathrm{Li}_k(x) = -\frac{1}{\Gamma(k)} \int_0^{\infty} \frac{s^{k-1}\, ds}{1 - x^{-1} e^s} \tag{5.11}$$

The equations, Eqs. (5.7)–(5.9), are nothing but Eqs. (1.1)–(1.3) and K_1, K_2, K_3 are model dependent parameters

$$K_1 = N(1+\mu) \sum_{s=1}^{\infty} \frac{(L - R(-\mu)^{s-1})}{1-(-\mu)^s} \left(\frac{\rho}{\rho-1}\right)^s s \tag{5.12}$$

$$K_2 = N^{-3/2} \frac{1+\mu}{\sqrt{2\pi}} \sum_{s=1}^{\infty} \frac{(L - R(-\mu)^{s-1})}{1-(-\mu)^s} \left(\frac{\rho}{\rho-1}\right)^s \frac{s^2(s-1+2\rho)}{((1-\rho)\,\rho)^{3/2}} \tag{5.13}$$

$$K_3 = N^{3/2} \sqrt{2\pi(1-\rho)\,\rho}. \tag{5.14}$$

The function $G(x)$ has been already obtained for the ASEP[11] and claimed to be universal for all models of KPZ universality class.[30] Simple form of the eigenvalue (5.7) allows one to define a general expression for the cumulants of the integrated particle current in scaling limit

$$\lim_{t\to\infty}\frac{\langle Y_t\rangle_c}{t}=K_1-K_2K_3=p(V_{\mathrm{AV}}-V_{\mathrm{ASEP}}) \tag{5.15}$$

$$\lim_{t\to\infty}\frac{\langle Y_t^n\rangle_c}{t}=K_2K_3^nG^{(n)}(0)=C_{\mathrm{AV}}^{(n)}+C_{\mathrm{ASEP}}^{(n)},\qquad n\geqslant 2, \tag{5.16}$$

where $G^{(n)}(0)$ is nth derivative of the function $G(x)$ at $x=0$. Here we divided the expressions for cumulants into two parts, which bring different physical content. Particularly, V_{AV} and V_{ASEP},

$$V_{\mathrm{AV}}\simeq\frac{L\mu+R}{(\rho_c-\rho)^2}\left[F_0(\rho,\rho_c)-\frac{F_1(\rho,\rho_c)}{N(\rho_c-\rho)^2}\right], \tag{5.17}$$

$$V_{\mathrm{ASEP}}\simeq L(1+\mu)(1-\rho)\left(1+\frac{1}{N}\right), \tag{5.18}$$

give the contributions to the average velocity coming from the avalanche part of the dynamics and its ASEP-like part respectively. The critical density, ρ_c, is defined as follows

$$\rho_c=\frac{1}{1+\mu},$$

which gives the phase diagram shown in Fig. 3, and the functions $F_0(\rho,\rho_c)$ and $F_1(\rho,\rho_c)$ are nonsingular below and at the critical point, $\rho=\rho_c$, so that the form of Eq. (5.17) explicitly shows critical singularities of the average velocity. While V_{ASEP} gives the average velocity of the ASEP when $\mu=-R/L$, V_{AV} vanishes at the same time. This is also the case for higher cumulants, which reproduce the results by Lee and Kim in this limit

$$C_{\mathrm{AV}}^{(n)}\simeq N^{\frac{3(n-1)}{2}}\frac{L\mu+R}{(\rho_c-\rho)^4}(2\pi\rho(1-\rho))^{\frac{n-1}{2}}\rho G^{(n)}(0)\,F_1(\rho,\rho_c), \tag{5.19}$$

$$C_{\mathrm{ASEP}}^{(n)}\simeq L(1+\mu)\,N^{\frac{3(n-1)}{2}}\frac{(2\pi\rho(1-\rho))^{\frac{n+1}{2}}}{2\pi}G^{(n)}(0). \tag{5.20}$$

Thus, when the density of particles, ρ, approaches its critical value, ρ_c, the divergency of average velocity of particles is characterized by the power law

$$V\sim V_{\mathrm{AV}}\sim(\rho_c-\rho)^{-\alpha} \tag{5.21}$$

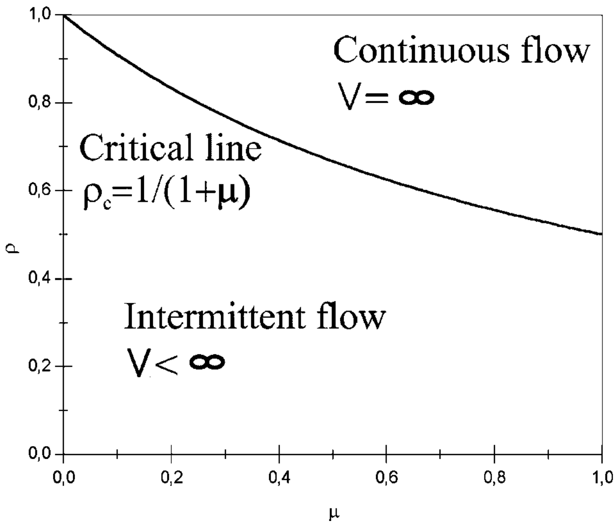

Fig. 3. The phase diagram of the asymmetric avalanche process.

with the critical exponent $\alpha = 2$. In ref. 22 this exponent has been shown to be nonuniversal with respect to the choice of different sets of dynamical rules μ_n. The other cumulants diverge as 4th power of $(\rho - \rho_c)$. The corrections to these laws are given in terms of the scaling variable, $N^{-1}(\rho - \rho_c)^{-2}$. The condition this variable is small defines the limits of applicability of the perturbative scheme and bounds the subcritical region.

$$|\rho - \rho_c| \gg 1/\sqrt{N}.$$

Closer to the critical line, we expect that the scaling will change as it did in the case $\mu = 0$. In addition, the case with arbitrary μ allows one to consider the region of phase diagram above the critical line, where the average velocity of particles will grow with N. This case requiring the modification of finite size expansion scheme is not considered here.

To obtain the cumulants, one needs knowing only the behavior of eigenvalue in the vicinity of the point $\gamma = 0$. At the same time, for the large deviation function, the whole range of γ is relevant. The solution considered above is valid for small negative γ. However, the definition of $G(x)$, Eqs. (5.8) and (5.9), maintains analyticity in the region $|C| < 1$. This implies

$$\mathrm{Li}_{3/2}(-1) < \gamma K_3 < \mathrm{Li}_{3/2}(1). \tag{5.22}$$

Beyond this domain one has to consider the solutions with different choice of the function $Z(x)$. Instead, we can directly use the analytical continuation of $G(x)$ proposed in ref. 11. To probe the whole range of negative γ, one can use the definition, Eqs. (5.8) and (5.9), where the functions $\mathrm{Li}_{3/2}(C)$, $\mathrm{Li}_{5/2}(C)$ are defined in the integral representation Eq. (5.11). For positive γ outside of the domain Eq. (5.22), we use the following expression

$$G(x) = \tfrac{8}{3}\sqrt{\pi}\,[-\ln(-C)]^{3/2} - \mathrm{Li}_{5/2}(-C) \tag{5.23}$$

$$x = 4\sqrt{\pi}\,[-\ln(-C)]^{1/2} + \mathrm{Li}_{3/2}(-C), \tag{5.24}$$

where $0 < C < -1$. Finally, using the definitions of $G(x)$ in different domains of γ, we get the large deviation function in the scaling limit

$$f(y) = K_3 H\left(\frac{y-K_1}{K_2K_3}\right) \tag{5.25}$$

where the universal function $H(x)$ is given by the following parametric expression

$$H(x) = G(\beta) - \beta G'(\beta) \tag{5.26}$$

$$x = G'(\beta). \tag{5.27}$$

In ref. 11 this function has been shown to be skew, i.e., to have different asymptotic behavior for its argument taking on a large negative or positive value. This is so for the ASEP, since the speeding up and slowing down are not equivalent due to the exclusion interaction. Specifically, it is much easier to slow down process by stopping or moving backward a single particle than to speed it up by moving forward all particles simultaneously. Similar qualitative interpretation of this asymmetry for the ASAP is also possible. One can see from the explicit form of Eqs. (5.15)–(5.18), that the leading terms of contributions to average velocity coming from the ASEP and avalanche parts of dynamics have different signs, the positive term corresponding to avalanches and the negative being for the ASEP drift. This is why the universal function $G(x)$ and subsequently $H(x)$ are different from it's standard ASEP form in the minus sign before its arguments, Eqs. (1.2) and (1.3) (compare with Eqs. (20) and (21) from ref. 11). Thus, in our case the speeding up and slowing down are interchanged comparing to the ASEP. Indeed, to initiate an avalanche one particle should move faster or slower then the others to reach eventually an occupied site. To prevent an avalanche all particles should move simultaneously in the same direction. The latter has much less probability then the former. However, comparing to the ASEP the situation is even more peculiar. The solution of the

ASEP is usually holds in the domain with definite direction of the drift, for example $L<R$. The opposite direction can be obtained by formal coordinate inversion $x\rightarrow -x$ that is equivalent to $L\leftrightarrow R$. The solution of the ASAP holds for all values of $L<1$ limited only by the condition $|\mu|<1$. Thus, for small densities, ρ, and the rate of the left driving, L, close to 1 the situation may take place when $V_{\mathrm{ASEP}}>V_{\mathrm{AV}}$, so that the resulting average velocity will be negative. This, however, does not affect the function $H(x)$ as its skewness is related only to the avalanche direction rather than to the Poissonian drift. At the same time full inversion transformation in the ASAP corresponds to simultaneous transformations $L\leftrightarrow R$ and $\mu\rightarrow 1/\mu$, which changes the direction of avalanches as well as that of Poissonian drift. At the same time, if one considers the fluctuations of the particle flow in the reference frame attached to the average flow of particles the macroscopic fluctuations are quite similar to the ASEP up to change of the distance scale

$$\frac{(Y_t-\langle Y_t\rangle)_{\mathrm{ASEP}}}{(Y_t-\langle Y_t\rangle)_{\mathrm{ASAP}}}$$
$$\rightarrow L(1+\mu)\left[1+\frac{\mu+R/L}{\mu}\sum_{s=1}^{\infty}\left(\frac{\rho}{\rho-1}\right)^s\frac{(-\mu)^s}{1-(-\mu)^s}\frac{s^2(s-1+2\rho)}{2((1-\rho)\,\rho)^2}\right]. \tag{5.28}$$

6. SUMMARY AND DISCUSSION

To conclude, we have considered the asymmetric avalanche process on the ring. To introduce the avalanche dynamics to the master equation for the Poissonian process, we used the technique of the recurrent boundary conditions. We have solved the master equation by the Bethe ansatz and studied the solution corresponding to the stationary state. As a result, we have calculated the cumulants of the integral particle current, exactly in the case $\mu=0$ and in the scaling limit for general μ. The large deviation function has been obtained, which has the structure, typical for models belonging to KPZ universality class. To calculate the finite size corrections, we used the modification of the perturbative scheme proposed by Kim in ref. 32. While in the leading orders the standard and modified approaches give the same results, it would be interesting to find out if this is so in arbitrary orders. Our investigation is valid for the densities below the critical point. However the model at the finite lattice can be considered for an arbitrary density of particles. One may expect that the scaling behavior of physical quantities should change above the critical point. The question of interest is how to modify the scheme to study the behavior of the model at the critical line and above.

APPENDIX A. SOLUTION OF THE BETHE EQUATIONS

First, we suppose that the roots of the Bethe equation are placed along the closed contour Γ encircling zero, and the solution still preserves the invariance with respect to complex conjugation like the solution corresponding to $\gamma \to 0$, Eq. (5.3). If we suppose also that the monovalued analytical function $Z^{-1}(\frac{p+1}{2N} - \frac{j}{N})$ can be defined everywhere at this contour, we obtain the mapping $j \to x_j$, which allow us to use the Euler–Maclaurin summation formula for transformation of the sum over the roots of Bethe equations into an integral along the segment $0 < j < p$ in the plane j with a correction term. The integral along the segment can be mapped into the integral along contour Γ in the plane x (see Appendix B)

$$\sum_{j=1}^{p} f(x_j) \to \int_1^p f(Z^{-1}(j))\, dj + \text{f.s.c.} \to \oint f(x)\, R(x)\, dx + \text{f.s.c.} \tag{A.1}$$

where f.s.c. is the correction term.

The functions of interest, e.g., the function $\Theta(y/x)$ can be represented on Γ as power series with additional logarithmic terms. In this case, the only correction term appears which corresponds to a contribution originating from the logarithm branch cut. As a result, the equation Eq. (5.6) simplifies to the following form

$$p_0(x) - \oint_\Gamma \Theta(y/x)\, R(y)\, dy - \gamma = i\pi\rho - 2\pi i \int_{x_0}^{x} R(x)\, dx. \tag{A.2}$$

The reference point x_0 is the cross point of the contour with the positive part of the real axis, which can be defined as $x_p e^{-2\pi i}$. The equation Eq. (A.2) coincides with that obtained in ref. 22 in the limit of infinite lattice after replacing the sum by the integral and neglecting the finite size corrections. It turns out that the solution of Eq. (A.2) gives the exact solution of Bethe equations, provided that the inverse function $Z^{-1}(\frac{p+1}{2N} - \frac{j}{N})$ is an analytical function in the segment $1 < j < p$, what should be checked afterwards from the solution obtained.

The only analytical solution of Eq. (A.2) corresponds to the case $\gamma = 0$. Taking into account the normalization

$$\oint R(x)\, dx = \rho. \tag{A.3}$$

we get the solution

$$R_0(x) = \frac{1}{2\pi i}\left(\frac{\rho}{x} + \frac{1}{1-x}\right). \tag{A.4}$$

According to Eq. (5.5) the function $Z(x)$ corresponding to this solution, can be obtained by the integrating of the density $R(x)$:

$$Z_0(x) = \frac{(p+1)}{2N} - \frac{1}{2\pi i}\left(\rho \ln \frac{x}{x_0} - \ln \frac{1-x}{1-x_0}\right)$$

Using the definition of $Z(x)$, Eq. (5.4), we obtain the equation for the roots x_j of the Bethe equations.

$$j = \frac{N}{2\pi i}\left(\rho \ln \frac{x_j}{x_0} - \ln \frac{1-x_j}{1-x_0}\right) \tag{A.5}$$

Considering j as a continuous parameter varying from 0 to p, Eq. (A.5) can be treated as an implicit definition of the contour Γ. It is easy to check that Eq. (A.5) has a solution corresponding to a closed contour around zero when x_0 varies in the interval $0 \leqslant x_0 \leqslant x_\rho$. The upper point x_ρ, defined by the equation $x_\rho^\rho - (1-x_\rho)\, \rho^\rho (1-\rho)^{1-\rho} = 0$, is a monotonous function of ρ changing in the range $0 \leqslant x_\rho \leqslant 1/2$ when ρ changes from 0 to 1. Every value of x_0 less then x_ρ corresponds to the particular position of the contour Γ passing through x_0. As it was shown above, in the case of finite p and $\gamma = 0$, all the roots are collapsed into one point $x = 0$. This situation is realized when $x_0 = 0$. The nonzero values of x_0 correspond to two limits $\gamma \to 0$, $p \to \infty$ taken simultaneously. One can see from the Eq. (5.3) that if we put $\gamma \sim \phi^p$, $(1 < \phi < 0)$, the value of x_0 takes different limits when $p \to \infty$ depending on ϕ. At last, the case $x_0 = x_\rho$ is realized when the decay of γ with growth of p is slower than exponential and particularly, when the limit $p \to \infty$ is taken for a fixed γ and then γ is put to zero. This case is of special interest for us because it gives the zero order of the solution we are looking for. When $x_0 > x_\rho$, the equation (A.5) has no solutions corresponding to a closed contour encircling the zero point.

Whereas the dependence of x_ρ on ρ is defined by the irrational equation Eq. (A.5), for $x_0 = x_\rho$, the connection between ρ and the cross point of Γ with the negative part of real axes, x_c is much simpler

$$x_c = \frac{\rho}{\rho - 1}. \tag{A.6}$$

Importance of this point lies in vanishing of the root density $R_0(x)$ at x_c:

$$R_0(x_c) = 0 \tag{A.7}$$

This fact is crucial for the Kim's perturbative scheme.[32, 33] Practically, it determines the range of applicability of Eq. (A.2). It has been noted above that Eq. (A.2) to be correct, the analytical function $Z^{-1}(\frac{p+1}{2N}-\frac{j}{N})$ should be defined to map the segment $0 \leqslant j < p$ to contour Γ. It is possible only if the first derivative $Z'(x) = -R(x)$ differs from zero everywhere in a given region.[39] According to Eq. (A.7), this is not the case at least at one point if $\gamma \neq 0$. Thus, we cannot apply the above arguments to whole contour Γ. To overcome this difficulty, we can separate Γ into two parts and apply the summation formula to intervals where the requirements of analyticity are satisfied. It seems natural that all troubles with analyticity are concentrated around the point of crossing the contour with the negative part of the real axis.

Consider the solution for odd p, and small positive γ. Then, the two roots closest to the negative part of the real axis, are conjugated to each other and can be denoted by

$$x_{(p-1)/2} = x_c e^{-i\epsilon}, \qquad x_{(p+1)/2} = x_c e^{i\epsilon}, \tag{A.8}$$

where x_c and ϵ are unknown real parameters. We exclude a small part of the contour Γ between $x_{(p-1)/2}$ and $x_{(p+1)/2}$ and assume the analyticity of $Z^{-1}(\frac{p+1}{2N}-\frac{j}{N})$ at the segments $1 < j < (p-1)/2$ and $(p+1)/2 < j < p$. This allows us to apply the Euler–Maclaurin formula for these segments separately. However, points $j = (p+1)/2$ and $j = (p-1)/2$ are located in the vicinity of a point on the real axis where $Z^{-1}(\frac{p+1}{2N}-\frac{j}{N})$ looses its analyticity. Therefore, the higher the order of derivatives of $Z^{-1}(\frac{p+1}{2N}-\frac{j}{N})$ at these points, the more singular their behavior. Practically, this means that all terms of the Euler–Maclaurin series have the same order in N. This is why an alternative variant of the summation formula in the Abel–Plana form has been applied in ref. 10. The Abel–Plana formula requires, however, the analyticity of $Z^{-1}(\frac{p+1}{2N}-\frac{j}{N})$ in the strips $1 < \operatorname{Re} j < (p-1)/2$ and $(p+1)/2 < \operatorname{Re} j < p$, which is assumed below.

Using the Abel–Plana formula, we can transform the sum over the roots of Bethe equations into two integrals along the segments $1 < j < (p-1)/2$ and $(p+1)/2 < j < p$ with a finite size correction term f.s.c. Then, these two integrals can be rewritten as the integral in the plane x along the closed contour Γ minus the integral along the small segment connecting the points $x_{(p-1)/2}$ and $x_{(p+1)/2}$.

$$\sum_{j=1}^{p} f(x_j) \to \int_1^{(p-1)/2} f(Z^{-1}(j))\, dj + \int_{(p+1)/2}^{p} f(Z^{-1}(j))\, dj + \text{f.s.c.}$$

$$\downarrow$$

$$\oint f(x)\, R(x)\, dx - \int_{x_{(p-1)/2}}^{x_{(p+1)/2}} f(y)\, R(y)\, dy + \text{f.s.c.}$$

To get the expression for the finite size correction term, we have to know the behavior of the function $Z^{-1}(\xi)$ in the vicinity of points $\xi=0, 1/N$. Once we know $Z(x)$, the inverse function can be constructed by inversion of its series at every point. However, the inverse function $Z^{-1}(\xi)$ become singular at the points $\xi=0, 1/N$ in the limit $N\to\infty$. This should be taken into account in constructing its series.

Consider the Taylor series of $Z(x)$ in points $x=x_c e^{\mp i\epsilon}$:

$$Z(x_c e^{\mp i\epsilon}+t)=\sum_{n=0}^{\infty}\frac{z_n^{\mp}}{n!}\,t^n$$

We can introduce new shifted variables $y=t-\sigma_{\mp}$ and consider the series expansion in y.

$$Z(x_c e^{\mp i\epsilon}-\sigma_{\mp}+y)=z_0^{\mp}+\delta^{\mp}+\sum_{n=1}^{\infty}\frac{b_n^{\mp}}{n!}\,y^n, \tag{A.9}$$

where

$$z_0^{-}=Z(x_c e^{-i\epsilon})=-\frac{1}{N};\qquad z_0^{+}=Z(x_c e^{i\epsilon})=0;$$

$$\delta_{\mp}=\sum_{n=1}^{\infty}\frac{z_n^{\mp}}{n!}\,\sigma_{\mp}^{n};\qquad b_n^{\mp}=\sum_{k=n}^{\infty}\frac{z_k^{\mp}\sigma_{\mp}^{k-n}}{(k-n)!}.$$

Two signs $\mp$, marking all parameters here are to remind that we consider two expansions around points $x_{(p-1)/2}=x_c e^{-i\epsilon}$ and $x_{(p+1)/2}=x_c e^{i\epsilon}$, and generally the parameters $z_k^-, \delta^-, b_n^-, \sigma_-$ are different from $z_k^+, \delta^+, b_n^+, \sigma_+$.

The shift of parameters $\sigma_{\mp}$ is defined by the condition

$$b_1^{\mp}=0 \tag{A.10}$$

that determines the shift of the expansion to the point where the density $R(x)$ is zero. Using the expansion (A.9), we can construct the inverse series

$$Z^{-1}\left(z_0^{\mp}+\frac{\xi}{N}\right)=\sum_{n=0}^{\infty}a_n\left(\frac{1}{iN}\right)^{\frac{n}{2}}\left(\pm i\sqrt{-\frac{\xi}{i}-\frac{\delta N}{i}}\right)^n$$

$$a_0=x_c e^{\mp i\epsilon}+\sigma,\qquad a_1=\sqrt{\frac{2}{b_2}},\qquad a_2=-\frac{b_3}{3b_2^2}, \tag{A.11}$$

$$a_3=\frac{1}{18\sqrt{2}}\left(\frac{1}{b_2}\right)^{\frac{7}{2}}(5b_3^2-3b_2b_4),\ldots$$

Here, we omitted the indices $\mp$ at variables a, b, σ, δ still implying two different expansions. To ensure the proper choice of a branch of the square root in both expansions, one can check if Eq. (A.11) for $\xi=0$ is satisfied. Finally, after some algebra (see Appendix C) we rewrite the Bethe equations in the following form:

$$\begin{aligned}R_s=&\frac{\theta(-s-1)}{2\pi i}-\frac{1}{\pi}\frac{(-\mu)^{|s|}}{1-(-\mu)^{|s|}}\left\{\frac{1}{4N}(\tau_+x_+^s-\tau_-x_-^s)+R_s\ln\frac{x_+}{x_-}\right.\\&+\frac{1}{2i}\sum_{n\neq s}\frac{R_n}{s-n}(x_+^{s-n}-x_-^{s-n})+\frac{1}{4N}(\tau_+x_+^s-\tau_-x_-^s)\\&+\frac{1}{4N}\sum_{n=1}^{\infty}\left(\frac{1}{2iN}\right)^{\frac{n}{2}}\frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}+1}}\left[c_{n,s}^-(\mathrm{Li}_{\frac{n}{2}+1}(-e^{-\pi\tau_-})-i^n\,\mathrm{Li}_{\frac{n}{2}+1}(-e^{\pi\tau_-}))\right.\\&\left.\left.-c_{n,s}^+(\mathrm{Li}_{\frac{n}{2}+1}(-e^{-\pi\tau_+})-(-i)^n\,\mathrm{Li}_{\frac{n}{2}+1}(-e^{\pi\tau_+}))\right]\right\}\end{aligned}\tag{A.12}$$

if $s\neq 0$ and

$$\epsilon R_0=-\sum_{s\neq 0,\,s=-\infty}^{\infty}R_s x_c^{-s}\frac{\sin\epsilon s}{s}+\frac{1}{2iN}\tag{A.13}$$

otherwise. For γ and ρ, we have respectively

$$\begin{aligned}\gamma=&\frac{1}{2Ni}\sum_{n=1}^{\infty}\left(\frac{1}{2iN}\right)^{\frac{n}{2}}\frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}+1}}\left[\bar{c}_n^-(\mathrm{Li}_{\frac{n}{2}+1}(-e^{-\pi\tau_-})-i^n\,\mathrm{Li}_{\frac{n}{2}+1}(-e^{\pi\tau_-}))\right.\\&\left.-\bar{c}_n^+(\mathrm{Li}_{\frac{n}{2}+1}(-e^{-\pi\tau_+})-(-i)^n\,\mathrm{Li}_{\frac{n}{2}+1}(-e^{\pi\tau_+}))\right]\\&+\frac{i}{2N}(\tau_-\ln x_--\tau_+\ln x_+)+\frac{R_0}{2}(\ln^2 x_--\ln^2 x_+)\\&+\sum_{n\neq 0}R_s\left(\frac{x_+^{-n}-x_-^{-n}}{n^2}+\frac{x_+^{-n}\ln x_+-x_-^{-n}\ln x_-}{n}\right)\end{aligned}\tag{A.14}$$

and

$$\rho=2\pi i R_0.\tag{A.15}$$

Here we introduced notations $\tau_\mp$, $x_\mp$, $c_{n,s}^\mp$ which are defined as follows

$$\delta^\mp=-\frac{i}{2N}(\mp i+\tau_\mp);\qquad x_\mp=x_c e^{\mp i\epsilon}+\sigma^\mp\tag{A.16}$$

$$\left[\sum_{n=0}^{\infty}a_n^\mp x^n\right]^s=\sum_{n=0}^{\infty}c_{n,s}^\mp x^n;\qquad \ln\left(\sum_{n=0}^{\infty}a_n^\mp x^n\right)=\sum_{n=0}^{\infty}\bar{c}_n^\mp x^n,\tag{A.17}$$

and R_s are the expansion coefficients of Laurent series

$$R(x)=\sum_{s=-\infty}^{\infty}\frac{R_s}{x^{s+1}}. \tag{A.18}$$

Using Eqs. (3.17), (B.7) together with Eqs. (C.1)–(C.4) we obtain the expression of eigenvalue in terms of R_s

$$\Lambda(\gamma)=2\pi i N\sum_{n=1}^{\infty}(-\mu)^{-n}\Lambda_n R_n, \tag{A.19}$$

where

$$\Lambda_n=(L-R(-\mu)^{n-1})(1+\mu)$$

While the equations Eqs. (A.12)–(A.15) look rather cumbersome, a significant simplification takes place if the following conditions are satisfied:

$$\tau_+=\tau_-=\tau; \tag{A.20a}$$

$$x_+=x_-=\tilde{x}; \tag{A.20b}$$

$$a_s^-=a_s^+=a_s. \tag{A.20c}$$

Then, instead of Eqs. (A.12), (A.14), we get

$$\begin{aligned} R_s=&\frac{\theta(-s-1)}{2\pi i}-\frac{1}{2\pi i}\frac{(-\mu)^{|s|}}{1-(-\mu)^{|s|}}\\ &\times\frac{1}{N^{3/2}}\sum_{n=0}^{\infty}\left(\frac{i}{2N}\right)^n\frac{\Gamma(n+\frac{3}{2})}{\pi^{n+\frac{3}{2}}}\frac{c_{2n+1,s}}{\sqrt{2i}}\operatorname{Li}_{n+\frac{3}{2}}(-e^{\pi\tau})\\ \gamma=&-\frac{1}{N^{3/2}}\sum_{n=0}^{\infty}\left(\frac{i}{2N}\right)^n\frac{\Gamma(n+\frac{3}{2})}{\pi^{n+\frac{3}{2}}}\frac{\bar{c}_{2n+1}}{\sqrt{2i}}\operatorname{Li}_{n+\frac{3}{2}}(-e^{\pi\tau}), \end{aligned} \tag{A.21}$$

which together with Eq. (A.19) reproduces the results of paper.[32] The conditions of Eqs. (A.20a) and (A.20b) are those accepted in ref. 32 as assumptions. Equation (A.20a) is equivalent to equality $\delta_+=\delta_-$ (see Eq. (32) in ref. 32). Equation (A.20b) is equivalent to the assumption that there is only one point $\tilde{x}$ where $Z'(\tilde{x})=0$ which is used as the expansion center in ref. 32. In our consideration, two complex conjugated points, x_+ and x_-, are possible that merge into one point $\tilde{x}$ in the limit $p\to\infty$. The third equality Eq. (A.20c) is a direct consequence of first two. Generally, there are no obvious reasons for Eqs. (A.20a) and (A.20b) to be satisfied.They can be checked a posteriori when the solution of Eqs. (A.12) and (A.14) is obtained. We checked them in first orders of the perturbative solution and found that they are correct in the first three orders which are necessary to reproduce the results of ref. 33.

To obtain the solution of the Eqs. (A.12)–(A.15) which is consistent with the exact solution in the case $\mu = 0$, one has to assume ϵ to behave as $\epsilon \sim N^{-1/2}$ when $N \to \infty$. Therefore, we assume the following expansion

$$\epsilon = \sum_{k=1}^{\infty} \frac{\epsilon_i}{N^{\frac{k}{2}}}. \tag{A.22}$$

The other values in the Eqs. (A.12)–(A.15) can be represented as similar expansions

$$R_s = \sum_{k=0}^{\infty} \frac{R_s^{(k)}}{N^{\frac{k}{2}}}, \qquad \rho = \sum_{k=0}^{\infty} \frac{\rho_i}{N^{\frac{k}{2}}}, \qquad \gamma = \sum_{k=3}^{\infty} \frac{\gamma_i}{N^{\frac{k}{2}}}. \tag{A.23}$$

Equation (A.4) is used as a zero order solution. Then Eqs. (A.12)–(A.15) should be solved order by order in powers of $N^{-1/2}$. The scaling dependence of γ corresponds to $\gamma N^{3/2} = \text{const}$. The limit $\gamma \to 0$ corresponds to the limit $\epsilon_1 \to 0$. The other parameters $\epsilon_2, \epsilon_3, \ldots$ depend on the way, how γ approaches zero when $N \to \infty$. However, the physical results do not depend on these parameters due to analyticity of eigenvalue. Solving Eqs. (A.12)–(A.15) in first four orders, we get the expression for eigenvalue given in Eqs. (5.7)–(5.9)

APPENDIX B. EVALUATION OF SUMS OVER ROOTS OF THE BETHE EQUATIONS

To evaluate the sum over roots of the Bethe equation, one can use the asymptotic formula approximating the sum by the integrals

$$\sum_{i=n}^{m} f(j) = \int_n^m f(j)\, dj + \mathrm{corr}(f, n, m) \tag{B.1}$$

with the correction term given by the asymptotic Euler–Maclaurin series

$$\mathrm{corr}(f, n, m) = \frac{1}{2}(f(m) + f(n)) + \sum_{i=2}^{\infty} \frac{B_i}{i!}(f^{(i-1)}(m) - f^{(i-1)}(n)) \tag{B.2}$$

or by the Abel–Plana integral form

$$\mathrm{corr}(f, n, m) = \frac{1}{2}(f(m) + f(n)) \tag{B.3}$$

$$+ \frac{1}{i} \int_0^\infty \frac{f(m+it) - f(m-it) - f(n+it) + f(n-it)}{e^{2\pi t} - 1} \, dt. \tag{B.4}$$

The former requires the analyticity of the function $f(j)$ at the segment of real axes $j \in [n, m]$ and the latter at the strip of complex plane, $\mathrm{Re}\, x \in [n, m]$. Let us suppose, that the analytical structure of the function $Z(x)$ allows one to define the analytical inverse function $Z^{-1}(\frac{p+1}{2N} - \frac{j}{N})$ that maps the segment $j \in [0, p]$ into the closed contour Γ encircling zero in the plane of the variable x. Then the derivatives with respect to j can be expressed in terms of x_j as follows

$$\frac{\partial}{\partial j} f(j) \to -\frac{1}{NR(x_j)} f'\left(-Z(x_j)\, N + \frac{p+1}{2}\right). \tag{B.5}$$

We are interested in calculation of sums of the form

$$\frac{1}{N} \sum_{j=1}^{p} F(x_j), \tag{B.6}$$

where $F(x)$ can be represented at the contour Γ as Laurent series with additional logarithmic term

$$F(x) = \bar{F} \ln x + \sum_{s=-\infty}^{\infty} F_s x^s. \tag{B.7}$$

The root x_p lies at the real part of positive axes. We can introduce its "twin" at the other side of logarithm branch cut, $x_0 = e^{-2\pi i} x_p$, which corresponds to $j = 0$. Then, application of Euler–Maclaurin formula Eq. (B.2) gives

$$\frac{1}{N} \sum_{j=1}^{p} F(x_j) = \frac{1}{N}\left(\sum_{j=0}^{p} F(x_j) - F(x_0)\right) = \oint_\Gamma F(y)\, R(y)\, dy + \frac{\pi i}{N} \bar{F} \tag{B.8}$$

Indeed, all derivatives in the Euler–Maclaurin series Eq. (B.2) taken at the points x_0 and x_p, being equal, are cancelled by each other. The only contribution to correction term comes from difference between imaginary parts of the logarithm at the banks of its branch cut. As it is shown in Section 5, this case is limited by $\gamma = 0$.

Let us consider the case when the roots $x_{(p-1)/2}$ and $x_{(p+1)/1}$ are located in the vicinity of the point x_c satisfying Eq. (A.7), and, therefore, we can not guarantee existence of an analytical function $Z^{-1}(\frac{p+1}{2N}-\frac{j}{N})$ that maps the segment $j\in[0,p]$ to closed contour Γ. We, however, still assume, that the mapping like this exists at two its segments, which connect points $x_0, x_{(p-1)/2}$ and $x_{(p+1)/1}, x_p$. One can see from Eq. (B.5), that every derivative with respect to j brings the competitive coefficients $1/N$ and $1/R(x_j)$, i.e., when N tends to infinity $R(x_{(p\pm1)/1})$ goes to zero. This is why we use Abel–Plana summation formula Eq. (B.3) instead of Eq. (B.2) to take into account all contributions of the same order in N. Applying it to each of two segments separately and using formula Eq. (A.8) for the roots $x_{(p-1)/2}$ and $x_{(p+1)/1}$ we obtain

$$\frac{1}{N}\sum_{j=1}^{p}F(x_j)=\sum_{s=-\infty}^{\infty}R_s(I_s^0+I_s^\epsilon)+\frac{\pi i}{N}\bar{F}$$
$$+\bar{F}\left(\frac{\ln x_c}{N}+\bar{\varkappa}\right)+\sum_{s=-\infty}^{\infty}F_s\left(\frac{x_c^s\cos\epsilon s}{N}+\frac{\varkappa_s}{N}\right) \tag{B.9}$$

where

$$I_s^0=2\pi i\begin{cases}F_s-\dfrac{\bar{F}}{sx_0^s} & s\neq 0\\ F_0+\bar{F}(\ln x_0+i\pi) & s=0,\end{cases} \tag{B.10}$$

$$I_s^\epsilon=-2i\epsilon\begin{cases}F_s+\displaystyle\sum_{n\neq s,\,n=-\infty}^{\infty}F_n\frac{x_c^{n-s}\sin(n-s)\,\epsilon}{\epsilon(n-s)} & \\ \quad+\dfrac{\bar{F}}{sx_c^s}\left(\left(\ln x_c+\dfrac{1}{s}\right)\dfrac{\sin\epsilon s}{\epsilon}-\cos\epsilon s\right) & s\neq 0\\ F_0+\displaystyle\sum_{n\neq 0,\,n=-\infty}^{\infty}F_n\frac{x_c^n\sin n\epsilon}{\epsilon n}+\bar{F}\ln x_c & s=0,\end{cases} \tag{B.11}$$

$$\varkappa_s=\frac{1}{i}\int_0^\infty\left\{\left[Z^{-1}\left(-\frac{1+it}{N}\right)\right]^s-\left[Z^{-1}\left(-\frac{1-it}{N}\right)\right]^s\right.$$
$$\left.-\left[Z^{-1}\left(-\frac{it}{N}\right)\right]^s+\left[Z^{-1}\left(\frac{it}{N}\right)\right]^s\right\}\Big/(e^{2\pi t}-1)\,dt, \tag{B.12}$$

$$\bar{\varkappa}=\frac{1}{i}\int_0^\infty\left\{\ln Z^{-1}\left(-\frac{1+it}{N}\right)-\ln Z^{-1}\left(-\frac{1-it}{N}\right)\right.$$
$$\left.-\ln Z^{-1}\left(-\frac{it}{N}\right)+\ln Z^{-1}\left(\frac{it}{N}\right)\right\}\Big/(e^{2\pi t}-1)\,dt \tag{B.13}$$

and R_s are the coefficients of the Laurent expansion of the density defined in Eq. (A.18).

APPENDIX C. DERIVATION OF EQUATIONS FOR R_s

Rewriting the sum in the Eq. (5.6) with the help of Eq. (B.9) and collecting coefficient of the same powers of x we get

$$s \neq 0: \qquad R_s = \frac{\theta(-s-1)}{2\pi i} - \frac{1}{\pi} \frac{(-\mu)^{|s|}}{1-(-\mu)^{|s|}}$$

$$\times \left(\epsilon R_s + \sum_{n \neq s,\, n=-\infty}^{\infty} \frac{x_c^{s-n}}{s-n} \sin \epsilon (s-n)\, R_n - \frac{x_c^s}{2iN} \cos \epsilon s - \frac{\varkappa_s}{2iN} \right) \tag{C.1}$$

$$s=0: \quad \epsilon R_0 = -\sum_{s \neq 0,\, s=-\infty}^{\infty} R_s x_c^{-s} \frac{\sin \epsilon s}{s} + \frac{1}{2iN} \tag{C.2}$$

$$\gamma = 2i \sum_{s \neq 0,\, s=-\infty}^{\infty} \frac{x_c^{-s}}{s} \left(\epsilon \cos \epsilon s - \frac{\sin \epsilon s}{s} \right) R_s + \frac{\bar{\varkappa}}{N} \tag{C.3}$$

$$\rho = 2\pi i R_0 \tag{C.4}$$

where we use the function $\Theta(y/x)$, treated as a function of the variable y, as the expansion valid at the contour Γ with coefficients defined like in Eq. (B.7)

$$\Theta_n = \begin{cases} \dfrac{(-\mu)^{|s|}}{s x^s} & s \neq 0 \\ \ln x & s = 0 \end{cases}; \qquad \bar{\Theta} = -1.$$

To go further one needs to obtain the explicit expressions for $\varkappa_s$, $\bar{\varkappa}$ in terms of R_s. Using the expansion of $Z^{-1}(z_0^{\mp} + \frac{\xi}{N})$, Eq. (A.11), we get

$$\varkappa_s = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2iN} \right)^{\frac{n}{2}} [c_{n,s}^{-} Y_n^{-} - c_{n,s}^{+} Y_n^{+}] \tag{C.5}$$

$$\bar{\varkappa} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1}{2iN} \right)^{\frac{n}{2}} [Y_n^{-} \bar{c}_n^{-} - Y_n^{+} \bar{c}_n^{+}], \tag{C.6}$$

where the coefficients $\bar{c}_n^{\mp}$ and $c_{n,s}^{\mp}$ are defined in Eq. (A.17) an the functions $Y_n^{\mp}$ are given by

$$Y_n^{\mp} = \frac{1}{i}\int_0^{\infty} \frac{[(\pm i\sqrt{-t+\tau_{\mp}-i})^n - (\pm i\sqrt{t+\tau_{\mp}-i})^n]}{e^{\pi t}-1}\,dt \tag{C.7}$$

The method of evaluation of these integrals is described in detail in ref. 33. Finally, for $Y_n^{\mp}$ we get

$$Y_n^{\mp} = \frac{1}{i}\left\{\left(\mp i - \frac{\tau_{\mp}\mp i}{\frac{n}{2}+1}\right)(\pm i\sqrt{\tau_{\mp}\mp i})^n + \frac{\Gamma(\frac{n}{2}+1)}{\pi^{\frac{n}{2}+1}}\left(\mathrm{Li}_{\frac{n}{2}+1}(-e^{-\pi\tau_{\mp}}) - (\pm i)^n\,\mathrm{Li}_{\frac{n}{2}+1}(-e^{\pi\tau_{\mp}})\right)\right\}. \tag{C.8}$$

Using the equalities

$$\sum_{n=1}^{\infty} c_{n,s}^{\mp}(\pm i\sqrt{\tau_{\mp}-i})^n\left(\frac{1}{2iN}\right)^{\frac{n}{2}} = [Z^{-1}(z_0^{\mp})]^s - c_{0,s}^{\mp}, \tag{C.9}$$

$$\sum_{n=1}^{\infty} \bar{c}_n^{\mp}(\pm i\sqrt{\tau_{\mp}-i})^n\left(\frac{1}{2iN}\right)^{\frac{n}{2}} = \ln[Z^{-1}(z_0^{\mp})] - \bar{c}_0^{\mp}, \tag{C.10}$$

$$\sum_{n=1}^{\infty} c_{n,s}^{\mp}(\pm i\sqrt{\tau_{\mp}-i})^n\left(\frac{1}{2iN}\right)^{\frac{n}{2}}\frac{\tau_{\mp}\mp i}{\frac{n}{2}+1} = \delta^{\mp}Nc_{0,s}^{\mp} + N\int_{c_{0,s}^{\mp}}^{Z^{-1}(z_0^{\mp})} x^s R(x)\,dx, \tag{C.11}$$

$$\sum_{n=1}^{\infty} \bar{c}_n^{\mp}(\pm i\sqrt{\tau_{\mp}-i})^n\left(\frac{1}{2iN}\right)^{\frac{n}{2}}\frac{\tau_{\mp}\mp i}{\frac{n}{2}+1} = N\delta^{\mp}\bar{c}_0^{\mp} + N\int_{c_{0,s}^{\mp}}^{Z^{-1}(z_0^{\mp})} \ln xR(x)\,dx, \tag{C.12}$$

$$Z^{-1}(z_0^{\mp}) = x_c e^{\mp i\epsilon}; \qquad c_{0,s}^{\mp} = x_{\mp}^s; \tag{C.13}$$

and Eqs. (A.16) we come to the system of equations for R_s, Eqs. (A.12)–(A.14).

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